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# Searching for integrable lattice maps using factorization

**Jarmo Hietarinta**[1] **and Claude Viallet**[2]

[1] Department of Physics, University of Turku, FIN-20014 Turku, Finland
[2] UMR 7589 Centre National de la Recherche Scientifique, Université Paris VI, Université Paris VII, Boîte 126, 4 Place Jussieu, F-75252 Paris Cedex 05, France

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**Abstract**
We analyse the factorization process for lattice maps, searching for integrable cases. The maps were assumed to be at most quadratic in the dependent variables, and we required minimal factorization (one linear factor) after two steps of iteration. The results were then classified using the algebraic entropy. Some new models with polynomial growth (strongly associated with integrability) were found. One of them is a nonsymmetric generalization of the homogeneous quadratic maps associated with KdV (modified and Schwarzian), for this new model we have also verified the 'consistency around a cube'.

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(Some figures in this article are in colour only in the electronic version)

## 1. Introduction

Although many interesting results have been obtained for discrete integrable systems and many properties have been clarified, we are still to a great extent in the 'taxonomic' stage of development. We do not have any clear classification and probably we have only discovered a small sample of integrable difference equations, the tip of the iceberg.

As with differential equations, there is no universal definition of integrability for discrete systems, and consensus about integrability can exist only within certain subclasses of equations. In this situation, 'integrability predictors' are very useful. These are the algorithmic methods based on a somewhat weaker definition of integrability, which nevertheless seem to be associated with integrable systems. For example, for a Hamiltonian system of $2N$ degrees of freedom, 'Liouville integrability' means the existence of $N$ independent and sufficiently regular functions that commute w.r.t. the Poisson bracket. Proof of integrability would then require the construction of the said commuting quantities, which is not algorithmic. In this case, a good algorithmic integrability predictor is the Painlevé test.

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### 1.1. Integrability test and concepts in 1D

For difference equations there is also a need for integrability predictors. One such method is 'singularity confinement' (SC) [1], which has been advocated as the discrete analogue of the Painlevé test. The idea of this test is to check what happens at a possible singularity of the evolution. Something special can only happen if the value of the dependent variable becomes infinite. A singularity is then defined as a point where the next step cannot be determined (for example, due to expressions like $\infty - \infty$). One then studies what happens near this point and the test condition is the following: if the dynamics lead to (or near) a singularity then after a few steps one should be able to get out of it, and this should take place *without essential loss of information*.

This principle has been used successfully in deriving new integrable difference equations, especially in finding discrete analogies of the Painlevé equations in [2] and in numerous subsequent papers by many authors, especially through the so-called de-autonomizing procedure. Despite its success, it turns out that passing the singularity confinement test is not sufficient for regularity, counterexamples are given in [3].

It turns out that singularity confinement is strongly associated with *reduced growth of complexity*. When one iterates a rational map, the expression becomes more and more complex as a rational expression of the initial values. If we measure this complexity by the degree of the numerator or denominator then generically the degree grows exponentially. However, the growth can be reduced if some common factors can be cancelled. Some amount of cancellation always happens when singularity is confined [4], this is why SC is such an efficient test. However, as far as integrability is concerned, it is the precise amount of cancellation that is crucial. This kind of complexity analysis, alias the algebraic entropy calculation, has turned out to be an unmatched integrability test for maps [3, 11]. The conjecture about growth after cancellations and integrability is as follows:

- growth is linear in $n \Rightarrow$ equation is linearizable.
- growth is polynomial in $n \Rightarrow$ equation is integrable.
- growth is exponential in $n \Rightarrow$ equation is chaotic.

### 1.2. Integrability tests and concepts in 2D

In principle, the idea of singularity confinement can be applied to lattice equations as well, this was already discussed in paper [1] where the test was introduced. More recently, this idea was applied in [5] where an 'ultra-local singularity confinement' was proposed (see also [6]). Nevertheless, it turns out that SC is rarely used in the study of 2D maps, perhaps due to the possibility of many different singularity confinement patterns depending on the arrangement of initial values. In contrast, the growth of complexity analysis can be easily applied to lattice equations as well (see [12, 14]), and we will use it here.

Perhaps the strongest form of integrability is 'consistency around a cube' as this kind of consistency immediately produces a Lax pair [7]. The idea here is that one should be able to consistently extend a two-dimensional map into three dimensions. If the 2D map is defined on an elementary square of the 2D lattice, then one constructs a cube with a suitably modified map on all sides of the cube (the modification deals with associating different parameters to different coordinate directions). This creates a potential consistency problem in the evolution as follows: suppose we are given the values at the corners $x_{000}, x_{100}, x_{010}, x_{001}$ (initial values), then the values at $x_{110}, x_{101}, x_{011}$ are uniquely determined by using the proper maps, but the value at $x_{111}$ can be computed in three different ways. The consistency test is that these three ways should all yield the same value. This is a type of the Bianchi identity. It has been used as a method to find and classify integrable lattice models, when associated with rather strong symmetry requirements and with [8] or without [9] the so-called tetrahedron property.

However, this kind of consistency is very sensitive on the role of *spectral parameters*: the coefficients of the model have specific roles and are interdependent. This means, for example, that the normal set of linear transformations would completely destroy any such association. Unfortunately, imposing this sort of interdependency adds enormous complications for a generic search program.

### *1.3. Plan of the paper*

We start from lattice relations defined on an elementary square of the 2D lattice. We assume that the relation is at most quadratic in the dependent variables. Due to the reasons discussed above, we have chosen factorization as the first selecting criterion: we impose as a factorization requirement that (at least) a linear factor can be extracted after two steps of evolution. Solving the resulting equations produces a list of maps, as described in section 3.

There are natural symmetries between the models. For example, two models may be related by a translation of the variables. Maps related by reflections or rotations can also be omitted, and this allows us to reduce the list to 80 cases.

We then analyse all the 80 cases with an algebraic entropy calculation, as explained in section 4. This yields a classification of the models according to their degree growth. The vanishing of entropy (polynomial growth) is the integrability detector we use here. However, since our search condition is just the factorization of one linear factor after two iterations, many models with exponential growth are still included, which may contain integrable models, when further constraints are introduced. Our final list contains just the different 'parents' without further analysis on their possible integrable 'descendents'.

In section 5, we analyse with some detail one particularly interesting multiparametric model, and show that it also verifies the condition of the consistency around the cube, thus proving its integrability. This model does not have the symmetries used in [8], and thus does not appear in the list given there, but as special cases it contains both the discrete modified and Schwarzian KdV.

## 2. Factorization in 2D

### *2.1. The map*

We consider maps defined on the Cartesian 2D lattice by relating the four corner values of an elementary square (see figure 1) with a multilinear relation:

$$\begin{aligned} &k x x_{[1]} x_{[2]} x_{[12]} + l_1 x x_{[1]} x_{[2]} + l_2 x x_{[1]} x_{[12]} + l_3 x x_{[2]} x_{[12]} + l_4 x_{[1]} x_{[2]} x_{[12]} \\ &\qquad + p_1 x x_{[1]} + p_2 x_{[1]} x_{[2]} + p_3 x_{[2]} x_{[12]} + p_4 x_{[12]} x + p_5 x x_{[2]} + p_6 x_{[1]} x_{[12]} \\ &\qquad + r_1 x + r_2 x_{[1]} + r_3 x_{[2]} + r_4 x_{[12]} + u \equiv Q(x, x_{[1]}, x_{[2]}, x_{[12]}; \alpha_1, \alpha_2) = 0. \qquad (1) \end{aligned}$$

Here $x_{n,m}$ is the dependent variable at a corner and we have used a shorthand notation, in which only the shifts with respect to the base point at lower left are indicated in a subscript in square brackets: $x_{n,m} = x_{00} = x$, $x_{n+1,m} = x_{10} = x_{[1]}$, $x_{n,m+1} = x_{01} = x_{[2]}$, $x_{n+1,m+1} = x_{11} = x_{[12]}$. (We will use indifferently these three notations.) As mentioned above, in the 3D consistency

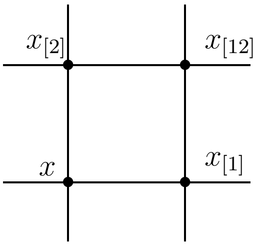


**Figure 1.** The map is defined on an elementary square of the 2D lattice.

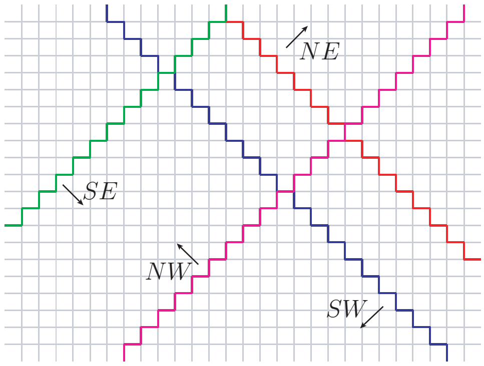


**Figure 2.** Fundamental evolutions on a square lattice.

approach, an important role is played by the spectral parameters $\alpha_s$, associated with specific directions, and they appear in the coefficients $k, l_i, p_i, r_i, u$. In this paper, however, these spectral parameters are ignored (except in section 5), since we only work on the 2D lattice.

Dynamics is defined by $Q(x_{n,m}, x_{n+1,m}, x_{n,m+1}, x_{n+1,m+1}) = 0$, using the multilinear $Q$ of (1), and this allows well-defined evolution from any staircase-like initial condition, up or down, since we can solve for any particular corner value in terms of the others, see figure 2.

The canonical examples of such maps are the following:

- Lattice KdV:

$$(p_1 - p_2 + x_{01} - x_{10})(p_1 + p_2 + x_{00} - x_{11}) = p_1^2 - p_2^2,$$

or after translation $x_{n,m} = u_{n,m} + p_1 n + p_2 m$

$$(u_{01} - u_{10})(u_{00} - u_{11}) = p_1^2 - p_2^2, \tag{2}$$

- Lattice MKdV:

$$p_1(x_{00}x_{01} - x_{10}x_{11}) = p_2(x_{00}x_{10} - x_{01}x_{11}), \tag{3}$$

- Lattice SKdV:

$$(x_{00} - x_{10})(x_{01} - x_{11})p_1^2 = (x_{00} - x_{01})(x_{10} - x_{11})p_2^2. \tag{4}$$

*2.2. Keeping track of factors*

When a rational map is iterated, some factors often get cancelled 'silently' in the resulting rational expression. This can be observed by comparing the degrees of the terms to the generic case. However, the best way to keep track of factors is to write the rational map as a polynomial map in a projective space. This was also the method that in the 1D case showed clearly the connection between singularity confinement and reduction in the growth of complexity [4]. In the 1D case, it turned out that the amount of cancellation had to be such that the degrees only grow polynomially ($\propto n^k$), while for nonintegrable systems we have exponential growth ($\rho^n$ with $\rho > 1$). The same happens for 2D systems [12, 14].

Recall that the map we are considering is given by a multilinear expression:

$$Q(x_{n,m}, x_{n+1,m}, x_{n,m+1}, x_{n+1,m+1}) = 0.$$

This equation can be homogenized by substituting $x_{n,m} = v_{n,m}/f_{n,m}$ and taking the numerator. We assume that $Q$ does not factor and that it depends on all the indicated variables. In the numerator of the homogenized $Q$, we isolate $v_{n+1,m+1}$ and $f_{n+1,m+1}$:

$$\begin{aligned} &A(v_{n,m}, v_{n+1,m}, v_{n,m+1}, f_{n,m}, f_{n+1,m}, f_{n,m+1})v_{n+1,m+1} \\ &\qquad + B(v_{n,m}, v_{n+1,m}, v_{n,m+1}, f_{n,m}, f_{n+1,m}, f_{n,m+1})f_{n+1,m+1} \end{aligned}$$

and then define the projective map as

$$\begin{cases} v_{n+1,m+1} = -B(v_{n,m}, v_{n+1,m}, v_{n,m+1}, f_{n,m}, f_{n+1,m}, f_{n,m+1}), \\ f_{n+1,m+1} = A(v_{n,m}, v_{n+1,m}, v_{n,m+1}, f_{n,m}, f_{n+1,m}, f_{n,m+1}). \end{cases} \tag{5}$$

The polynomials $A$ and $B$ are both of degree 3 in the indicated variables, and they cannot have common factors, since $Q$ was assumed irreducible.

For practical reasons, we consider in this paper *only the quadratic* maps, i.e., in (1) we take $k = 0$, $l_i = 0$. In that case the projective map is given by

$$\begin{cases} v_{[12]} = -[p_1 v v_{[1]} f_{[2]} + p_2 v_{[1]} v_{[2]} f + p_5 v v_{[2]} f_{[1]} + r_1 v f_{[1]} f_{[2]} + r_2 v_{[1]} f_{[2]} f \\ \quad + r_3 v_{[2]} f_{[1]} f + u f f_{[1]} f_{[2]}], \\ f_{[12]} = p_3 v_{[2]} f_{[1]} f + p_4 v f_{[1]} f_{[2]} + p_6 v_{[1]} f_{[2]} f + r_4 f f_{[1]} f_{[2]}. \end{cases} \tag{6}$$

From this we see immediately that the default degree growth is

$$\deg(z_{n+1,m+1}) = \deg(z_{n+1,m}) + \deg(z_{n,m+1}) + \deg(z_{n,m}), \tag{7}$$

where $z = v$ or $f$, since they have the same degree.

For the initial values, we may take any staircase-like configuration and in the rational representation take arbitrary $x$'s at each point. In the projective representation we may also take arbitrary $v$'s at all initial points, but we should use the same $f$, because projectivity only adds one free overall factor. In the quadratic case, this means the cancellation of one extra term at each step, so that the sequence of maximal degrees (without any factorization) is $1, 2, 4, 9, 21, \ldots$, which corresponds to the asymptotically exponential growth $(1 + \sqrt{2})^k$.

For the search part of our factorization study, we will only consider the first few steps in the iteration of (6). In principle, two different initial configurations are often used, staircase and corner, and the default degrees in these two cases are given in figure 3. The interesting factorization, that can be used to predict integrability, happens at the point where the default degree shown in figure 3 is 9 or 7, respectively. In the search part, we use the corner

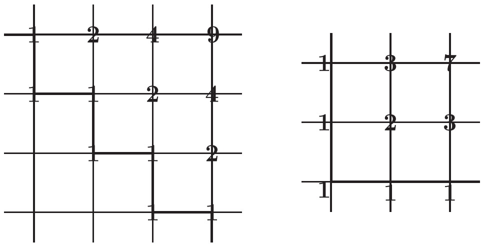


**Figure 3.** Maximum degrees with staircase and corner initial states for the quadratic relations.

configuration because the total degree is then the smallest. In the degree growth analysis, we have used the staircase configuration.

### *2.3. Factorization of known models*

Consider the discrete KdV model given by

$$(x_{n,m+1} - x_{n+1,m})(x_{n,m} - x_{n+1,m+1}) = a,$$

the homogenized $Q$ is now

$$\begin{aligned}Q = {} & f_{n,m+1} f_{n,m} v_{n+1,m+1} v_{n+1,m} - f_{n+1,m} f_{n,m} v_{n+1,m+1} v_{n,m+1} - f_{n+1,m+1} f_{n,m+1} v_{n+1,m} v_{n,m} \\ & + f_{n+1,m+1} f_{n+1,m} v_{n,m+1} v_{n,m} - f_{n+1,m+1} f_{n+1,m} f_{n,m+1} f_{n,m} a,\end{aligned}$$

and the projective map

$$\begin{cases} v_{n+1,m+1} = f_{n,m+1} v_{n+1,m} v_{n,m} - f_{n+1,m} v_{n,m+1} v_{n,m} + f_{n+1,m} f_{n,m+1} f_{n,m} a, \\ f_{n+1,m+1} = f_{n,m+1} f_{n,m} v_{n+1,m} - f_{n+1,m} f_{n,m} v_{n,m+1}. \end{cases} \tag{8}$$

In the corner case, we take $f_{0,m} = f_{n,0} = f_{00}$ and after cancelling one $f_{00}$ at every step one finds the first interesting GCD (greatest common divisor of the list of polynomials) at (2, 2):

$$\mathrm{GCD}(v_{22}, f_{22}) = (v_{01} - v_{10})^2.$$

Exactly the same GCD is found from the stair configuration, with initial values restricted by $f_{n,m} = f_{00}$ for $n+m = 0, 1$. The remaining (different) factors of $v_{22}$, $f_{22}$ are rather lengthy, in the corner case they are of degree 5 in the initial values and have 23 and 13 terms, respectively, and for the staircase initial configuration the degree is 7 and the number of terms are 112 and 76, respectively.

For the simplest linear model $x_{[12]} + x_{[1]} + x_{[2]} + x = 0$, the cancellations are so strong that the map stays linear, similarly for the quadratic model $x_{[12]}x + ax_{[1]}x_{[2]} = 0$, the projective map stays quadratic.

In [8], a list of lattice models having the 'consistency around a cube' property was given. The divisors for the quadratic cases, using corner configuration, are as follows:

(H1) map: $(x_{00} - x_{11})(x_{10} - x_{01}) + \beta - \alpha = 0$,
factor: $(v_{10} - v_{01})^2$.

(H2) map: $(x_{00} - x_{11})(x_{10} - x_{01}) + (\beta - \alpha)(x_{00} + x_{10} + x_{01} + x_{11}) + \beta^2 - \alpha^2 = 0$,
factor: $(v_{10} - v_{01} + (\alpha - \beta) f_{00})(v_{10} - v_{01} - (\alpha - \beta) f_{00})$.

(H3) map: $\alpha(x_{00}x_{10} + x_{01}x_{11}) - \beta(x_{00}x_{01} + x_{10}x_{11}) + \delta(\alpha^2 - \beta^2) = 0$,
factor: $(v_{10}\alpha - v_{01}\beta)(v_{01}\alpha - v_{10}\beta)$.
(A1) map: $\alpha(x_{00} + x_{01})(x_{10} + x_{11}) - \beta(x_{00} + x_{10})(x_{01} + x_{11}) - \delta^2\alpha\beta(\alpha - \beta) = 0$,
factor: $(v_{10} - v_{01} + \delta(\alpha - \beta)f_{00})(v_{10} - v_{01} - \delta(\alpha - \beta)f_{00})$.
(Q1) map: $\alpha(x_{00} - x_{01})(x_{10} - x_{11}) - \beta(x_{00} - x_{10})(x_{01} - x_{11}) + \delta^2\alpha\beta(\alpha - \beta) = 0$,
factor: $(v_{10} - v_{01} + \delta(\alpha - \beta)f_{00})(v_{10} - v_{01} - \delta(\alpha - \beta)f_{00})$.
(Q2) map: $\alpha(x_{00} - x_{01})(x_{10} - x_{11}) - \beta(x_{00} - x_{10})(x_{01} - x_{11})$
$+ \alpha\beta(\alpha - \beta)(x_{00} + x_{10} + x_{01} + x_{11}) - \alpha\beta(\alpha - \beta)(\alpha^2 - \alpha\beta + \beta^2) = 0$,
factor: $(v_{10} - v_{01})^2 - 2(\alpha - \beta)^2 f_{00}(v_{10} + v_{01}) + (\alpha - \beta)^4 f_{00}^2$.
(Q3) map: $(\beta^2 - \alpha^2)(x_{00}x_{11} + x_{10}x_{01}) + \beta(\alpha^2 - 1)(x_{00}x_{10} + x_{01}x_{11})$
$- \alpha(\beta^2 - 1)(x_{00}x_{01} + x_{10}x_{11}) - \delta^2(\alpha^2 - \beta^2)(\alpha^2 - 1)(\beta^2 - 1)/(4\alpha\beta) = 0$,
factor: $4\alpha\beta(v_{01}\alpha - v_{10}\beta)(v_{01}\beta - v_{10}\alpha) + \delta^2(\alpha^2 - \beta^2)^2 f_{00}^2$.

Note that for H1–H3, A1, Q1 we get two linear factors while for Q2, Q3 the quadratic factor is irreducible (if $\delta = 0$ then the divisor for Q3 does factor). Note also that the common factors can be used to study relationships between maps; for the above list, we just observe that A1 is obtained from Q1 with $x_{nm} \mapsto (-1)^{n+m}x_{nm}$.

## 3. Search

### *3.1. The method*

We have seen above that in all the known integrable quadratic lattice maps, the result at (2, 2) factorizes with a quadratic GCD (with the exception of linearizable models that may have even more factors). In many models this quadratic common divisor factorizes, but not always. The search for a linear factor is computationally easier, and therefore this search is restricted to that, but hope to return to a search for model with irreducible quadratic factors later. The possibility of irreducible factors of degree higher than 2 is open, and no examples are known.

In this search project, we use the corner configuration because computations are then simpler. Also, for computational simplicity, we use $x$ rather than $v$, $f$, but to prevent accidental factorizations we proceed as follows: we calculate $x_{11}, x_{21}, x_{12}$ using a *generic* $Q$ (which does not factorize) and substitute the obtained values into the equation $Q(x_{11}, x_{12}, x_{21}, x_{22}) = 0$ and take its numerator (to be called $Q_N$ from now on). We then require that this $Q_N$ factorizes as $Q_N =$ (some polynomial in $x_{00}, x_{10}, x_{01}, x_{20}, x_{02}, x_{22}$) $\times$ $[x_{01} + s_{10}x_{10} + s_{00}]$. Here, we may assume that $x_{01}$ has unit coefficient because the other coefficients $s$ can be rational in $x_{00}$; furthermore, if $x_{01}$ were zero but $x_{10}$ not, we could use $n \leftrightarrow m$ reflection.

If we keep track of all the other variables except $x_{00}$, we have generically

$$Q_N = \sum_{\alpha,\mu,\nu=0}^{1} \sum_{t,v=0}^{4} x_{22}^{\alpha} x_{20}^{\mu} x_{02}^{\nu} x_{10}^{t} x_{01}^{v} g(x_{00}, \alpha, \nu, \mu, t, v) \tag{9}$$

for some polynomial $g$, furthermore we know something about the exponents: $\alpha + \mu + \nu + t + v \leqslant 7$. Because of this observation, we can make the following ansatz for the factorization $Q = PS$:

$$S = x_{01} + s_{10}x_{10} + s_{00}, \tag{10}$$

$$P = \sum_{\alpha,\mu,\nu=0}^{1} \sum_{t,v=0}^{4} x_{22}^{\alpha} x_{20}^{\mu} x_{02}^{\nu} x_{10}^{t} x_{01}^{v} d(x_{00}, \alpha, \nu, \mu, t, v), \tag{11}$$

where $\alpha + \mu + \nu + t + v \leqslant 6$ and $d, s$ are rational in $x_{00}$.

Given the ansatz (1) with the quadratic restriction $k = 0, l_i = 0$, we compute the $Q_N$ (which has 15 966 terms) and subtract the ansatz $PS$ above. The equations are then formed by taking the coefficients of different powers of $x_{22}, x_{20}, x_{02}, x_{10}, x_{01}$ with fixed ordering. Next we determined the 134 functions $d$ in $P$ by considering the leading terms of $Q - P(x_{01} + \cdots) = 0$. This is fully automatized using REDUCE [10]. The remaining equations were then solved for the functions $s$ and for the parameters of the map itself, $p_i, r_i, u$. The solution process branched a lot and required numerous separate computing sessions.

## 4. Results

If the computations led to a situation where the resulting map (a) factorized or (b) did not depend on all corners, that branch was terminated immediately, but even then we got 125 'raw' results. From this set, we omitted the maps that could be obtained by rotation or reflection from other maps (using translation by a constant if needed), this reduced the number from 125 to 80. For all those remaining models, we have calculated the algebraic entropy as explained in the following section.

### *4.1. Growth patterns and the algebraic entropy computations*

Suppose initial data are given on a line which allows the determination of the values at all points of the lattice, for example, a regular diagonal staircase. The multilinear relation (1) allows us to define evolutions in the following way: we iterate the relation by calculating the values on diagonals moving away from the initial staircase, as in the previous section. After cancellations we get a sequence of degrees $d_n$.

We may in this way define four fundamental evolutions, corresponding to the initial data given on the diagonals with slope +1 or −1, and evolutions towards the four corners of the lattice. We denote them 'NE, SE, SW, NW' by the orientation of the evolutions (towards North-East, South-East and so on), see figure 2. To each evolution, we associate an entropy with the definition inspired by the one-dimensional case [3, 11, 12, 14]:

$$\epsilon = \lim_{n\to\infty} \frac{1}{n} \log(d_n). \tag{12}$$

These entropies always exist [11] because of the subadditivity property of the logarithm of the degree of composed maps.

A full calculation of iterates is usually beyond reach. We can however get explicit sequences of degrees as explained in [12].

Suppose we start from initial values distributed on a diagonal, containing $\nu$ vertices $V_1, \ldots, V_\nu$. For each of these $\nu$ vertices, we assign to the dependent variable $x_{n,m}$ an initial value of the form

$$x_{[V_k]} = \frac{\alpha_k + \beta_k t}{\alpha_0 + \beta_0 t}, \qquad k = 1, \ldots, \nu, \tag{13}$$

where $\alpha_0, \beta_0$ and $\alpha_k, \beta_k$ ($k = 1, \ldots, \nu$) are arbitrary constants, and $t$ is some unknown. We then calculate the values of $x$ at the vertices which are within the range of our set of initial data. These values are rational fractions of $t$, whose numerator and denominator are of the same degree in $t$, and that is the degree we are looking for.

The next step is then to obtain a 'degree growth', i.e., to extract the value of the entropy from the first few terms of the sequence $\{d_n\}$. One very fruitful method is to introduce the generating function of the sequence of degrees:

$$g(s) = \sum_{k=0}^{\infty} s^k d_k \tag{14}$$

and try to fit it with a rational fraction. The remarkable fact is that this works surprisingly well, as it did for maps, although we know that it may not always be the case [13]. This means that we can often calculate the asymptotic behaviour measured by (12) from a finite beginning part of the sequence of degrees.

We have done this calculation for all four evolutions of the 80 cases we got. The $4 \times 80$ calculations resulted with a number of different sequences of degrees, which could all be fitted with the rational generating functions. Some have a linear growth, some are quadratic and some have an exponential growth. The sequences are given in the following tables, containing the beginning of the sequences, the generating function and the numerical value $\rho$ of the growth $\rho^n$:

Linear growth:

| | Sequence of degrees | Generating function | Growth |
|---|---|---|---|
| $l_1$ | 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, ... | $\frac{1}{(1-s)^2}$ | 1 |
| $l_2$ | 1, 2, 3, 5, 6, 8, 9, 11, 12, 14, ... | $\frac{1+s+s^3}{(s+1)(1-s)^2}$ | 1 |

Quadratic growth:

| | Sequence of degrees | Generating function | Growth |
|---|---|---|---|
| $q_1$ | 1, 2, 3, 5, 7, 10, 13, 17, 21, 26, ... | $\frac{1-s^2+s^3}{(s+1)(1-s)^3}$ | 1 |
| $q_2$ | 1, 2, 4, 6, 9, 12, 16, 20, 25, 30, ... | $\frac{1}{(s+1)(1-s)^3}$ | 1 |
| $q_3$ | 1, 2, 3, 5, 7, 11, 14, 20, 24, 32, ... | $\frac{1+s-s^2+s^4+s^5}{(s+1)^2(1-s)^3}$ | 1 |
| $q_4$ | 1, 2, 3, 5, 8, 12, 17, 23, 30, 38, ... | $\frac{1-s+s^3}{(1-s)^3}$ | 1 |
| $q_5$ | 1, 2, 4, 7, 11, 16, 22, 29, 37, 46, ... | $\frac{1-s+s^2}{(1-s)^3}$ | 1 |

Exponential growth:

| | Sequence of degrees | Generating function | Growth |
|---|---|---|---|
| $e_1$ | 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ... | $\frac{1+s}{1-s-s^2}$ | 1.618 |
| $e_2$ | 1, 2, 4, 7, 12, 20, 33, 54, 88, 143, ... | $\frac{1}{(1-s)(1-s-s^2)}$ | 1.618 |
| $e_3$ | 1, 2, 3, 5, 9, 16, 28, 49, 86, 151, ... | $\frac{1}{1-2s+s^2-s^3}$ | 1.755 |
| $e_4$ | 1, 2, 3, 5, 9, 17, 32, 60, 112, 209, ... | $\frac{(1-s)(s+1)}{1-2s+s^3-s^4}$ | 1.867 |
| $e_5$ | 1, 2, 4, 7, 13, 24, 45, 84, 157, 293, ... | $\frac{1}{1-2s+s^3-s^4}$ | 1.867 |
| $e_6$ | 1, 2, 4, 8, 15, 28, 52, 97, 181, 338, ... | $\frac{(1+s)(1-s+s^2)}{1-2s+s^3-s^4}$ | 1.867 |
| $e_7$ | 1, 2, 3, 5, 9, 17, 33, 65, 129, 257, ... | $\frac{1-s-s^2}{(1-2s)(1-s)}$ | 2.0 |
| $e_8$ | 1, 1, 3, 5, 11, 21, 43, 85, 171, 341, ... | $\frac{1}{(1+s)(1-2s)}$ | 2.0 |
| $e_9$ | 1, 1, 3, 6, 12, 24, 48, 96, 192, 384, ... | $\frac{1-s+s^2}{1-2s}$ | 2.0 |
| $e_{10}$ | 1, 2, 4, 7, 14, 27, 54, 107, 214, 427, ... | $\frac{1-s^2-s^3}{(1-s)(1-2s)(1+s)}$ | 2.0 |
| $e_{11}$ | 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, ... | $\frac{1}{1-2s}$ | 2.0 |
| $e_{12}$ | 1, 2, 4, 8, 16, 32, 65, 133, 274, 566, ... | $\frac{(1+s)(1-s)^2}{1-3s+s^2+3s^3-2s^4-s^6}$ | 2.067 |
| $e_{13}$ | 1, 2, 4, 8, 16, 33, 68, 141, 292, 605, ... | $\frac{(1-s)(s+1)}{1-2s-s^2+2s^3-s^5}$ | 2.071 |
| $e_{14}$ | 1, 2, 4, 8, 16, 33, 69, 145, 305, 642, ... | $\frac{1-s-s^4}{(1-s)(1-2s-s^4)}$ | 2.107 |
| $e_{15}$ | 1, 2, 4, 8, 17, 37, 82, 183, 410, 920, ... | $\frac{1-s-s^2}{(1-s)(1-2s-s^2+s^3)}$ | 2.247 |
| $e_{16}$ | 1, 2, 4, 9, 20, 45, 101, 227, 510, 1146, ... | $\frac{(1-s)(s+1)}{1-2s-s^2+s^3}$ | 2.247 |
| $e_*$ | 1, 2, 4, 9, 21, 50, 120, 289, 697, 1682, ... | $\frac{1-s-s^2}{(1-s)(1-2s-s^2)}$ | 2.414 |

In the previous table, the value $e_*$ is given for reference, it corresponds to the absence of factorization and is an upper bound. The factorization condition we have used is a rather mild one, it says that the fourth number should be 8 or less. (The pattern $e_{16}$ does not have this cancellation, it arises in a model, which has cancellations in two direction, but not in the other two.)

From these tables we observe that sequences with asymptotically exponential growth sometimes start as slowly as those with linear growth: compare for example patterns of $e_1, e_3, e_4, e_7, e_8$ with $l_2$.

### *4.2. Classification into parents and descendents*

We are not going to give the full listing of the 80 models obtained because the solution method is insensitive to subcase dependency. The criterion was only that there is one linear factor at position (2, 2) and this does not yet guarantee the integrability. It is possible that one model is obtained from another one by specializing the values of the parameters, giving a notion of descendent. This process provides a partial ordering of the models, the descendents of a model having a smaller entropy. It may happen that a model with nonvanishing entropy, thus *a priori* nonintegrable, has a descendent with vanishing entropy.

The list that is presented next is comprehensive in the sense that any integrable case with a linear factor at (2, 2) will be a subcase of one of the presented models. The list is rather short and therefore further studies of subcases with more factorization will not be overwhelming, but not included here.

The classification is up to transformations of the type

$$x_{nm} \mapsto x_{nm} + a(n - n_0) + b(m - m_0) + c,$$

where the $a, b$ terms can only be used if the result is $n, m$ independent. We would also like to note that innocuous reparameterizations can change the factorization properties because computer factorization is normally over integers. Examples of this can be seen in cases 2 and 5.

The ancestral models are as follows:

*Case 1.* This is a homogeneous model, where one quadratic term is missing, all others have arbitrary coefficients:

$$x_{00}x_{10}p_1 + x_{00}x_{11}p_4 + x_{10}x_{01}p_2 + x_{10}x_{11}p_6 + x_{01}x_{11}p_3 = 0. \tag{15}$$

The degree sequences in the four directions are $e_{15}, e_{16}, e_{16}, e_{15}$. The missing term above is $x_{00}x_{01}$, but it could be rotated to any other side of the elementary square. Since the model is not rotationally symmetric, it can have different growth patterns in different directions. The pattern $e_{16}$ corresponds to growth without any cancellations at (2, 2), but $e_{15}$, which has some cancellations, is obtained in the direction of the test. As for its subcases, we just mention that if $p_3 = 0$, i.e., the opposite side $x_{10}x_{11}$ is also missing, then the map is linearizable.

*Case 2.* This is also homogeneous, but it has 'symmetric cross' $x_{00}x_{11} + x_{10}x_{01}$ and all other coefficients are arbitrary:

$$\begin{aligned}x_{00}x_{10}p_1 + x_{00}x_{01}p_5(p_1p_3 + p_2) + (x_{00}x_{11} + x_{10}x_{01})p_2 + x_{10}x_{11}p_6 \\ + x_{01}x_{11}p_3(p_5p_6 - p_2) = 0.\end{aligned} \tag{16}$$

This has quadratic growth $q_5$ in all directions. This result illustrated the role of parameterization in factorization: there are several ways to reparameterize $p_3, p_5$ so that combinations $p_5(p_1p_3 + p_2)$ and $p_3(p_5p_6 - p_2)$ look simpler, but this could easily lead to one quadratic rather than two linear factors at the corner.

The next four models have a free nonhomogeneous constant term.

*Case 3.* Here there are completely arbitrary linear and constant terms, while in the quadratic part two opposite sides are missing ($x_{00}x_{10}$ and $x_{01}x_{11}$):

$$x_{00}x_{01}p_5 + x_{00}x_{11}p_4 + x_{10}x_{01}p_2 + x_{10}x_{11}p_6 + x_{00}r_1 + x_{10}r_2 + x_{01}r_3 + x_{11}r_4 + u = 0. \tag{17}$$

The growth pattern is $e_{11}$ in all directions, so as it stands the model is not integrable. Among its subcases we would like to mention

$$(p_3x_{01}+x_{00})(p_3x_{11}+x_{10})+(p_3x_{01}+x_{10})r_3=0 \tag{18}$$

and

$$(x_{00}-x_{01})(x_{10}-x_{11})+(x_{00}-x_{11})r_4+(x_{01}-x_{10})r_3+u=0 \tag{19}$$

both integrable, with growth patterns $q_3$ in all directions.

*Case 4.* Here in the homogeneous part the cross is missing, and other terms have arbitrary coefficients, except that there is a relation between the $r_i$ coefficients:

$$\begin{aligned} &x_{11}x_{10}p_6+x_{11}x_{01}p_3+x_{10}x_{00}p_1+x_{01}x_{00}p_5+x_{11}p_3p_6r_4+x_{10}p_6r_2+x_{01}p_3r_3\\ &\qquad +x_{00}(-p_1p_5r_4+p_1r_3+p_5r_2)+u=0. \end{aligned} \tag{20}$$

The parameters have been scaled to simplify the model, but still nicer forms of writing it may exist. The growth pattern is again $e_{11}$ in all directions. This contains as subcases the following integrable ones:

$$(x_{00}x_{01}+x_{10}x_{11})p_5+(x_{00}x_{10}+x_{01}x_{11})p_3+u=0, \tag{21}$$

$$(x_{00}-x_{11})(x_{01}-x_{10})+r_1(x_{00}+x_{10}+x_{01}+x_{11})+u=0, \tag{22}$$

which include H1–H3 of [8].

*Case 5.* The following model is integrable, with pattern $q_5$ in all directions:

$$(x_{11}+x_{00})(x_{10}+x_{01})p_2+(x_{11}+x_{01})(x_{00}+x_{10})p_3+p_3p_2(p_3+p_2)u^2=0, \tag{23}$$

or in another way of writing

$$\begin{aligned} &(x_{11}x_{10}+x_{01}x_{00})(p_2+p_3)+(x_{11}x_{01}+x_{10}x_{00})p_3+(x_{11}x_{00}+x_{10}x_{01})p_2\\ &\qquad +p_3p_2(p_3+p_2)u^2=0. \end{aligned} \tag{24}$$

The above parameterization of the constant term is necessary in order to have two linear factors at $(2,2)$, otherwise we would get one quadratic factor. This case becomes Q1 with scaling $x_{nm}\mapsto(-1)^{n-n_0}x_{nm}$ (it is not separately listed in [8], but the situation is similar to A1).

*Case 6*

$$\begin{aligned} &x_{11}x_{00}+x_{10}x_{01}+(x_{11}x_{01}+x_{10}x_{00})p_3-(x_{11}x_{10}+x_{01}x_{00})(p_3+1)\\ &\qquad +(x_{11}-x_{00})r_4+(x_{10}-x_{01})r_2-(u(p_3+1)+r_4)(up_3+r_4)+ur_2=0, \end{aligned} \tag{25}$$

which is integrable with growth $q_5$ in all directions. If $r_4=r_2=0$ we get Q1 of [8]. In fact, if $p_3\neq 0,-1$, we can apply an $n,m$-dependent translation to put $r_2=r_4=0$.

The remaining maps have linear and constant terms, but the constant term depends on the linear terms. There are also interesting relationships between the parameters of the homogeneous terms. There may be other interesting representations obtainable with the transformation $x_{ij}\to\mu x_{ij}+\nu$.

*Case 7*

$$\begin{aligned} &x_{11}x_{10}p_3^2+x_{01}x_{00}p_6^2+x_{11}x_{01}p_3p_1+x_{10}x_{00}p_3p_1^{-1}p_6^2\\ &\qquad +(x_{11}p_3+x_{10}p_3+x_{01}p_6+x_{00}p_6)r_1+r_1^2=0 \end{aligned} \tag{26}$$

with growth $e_{11}$ in all directions.

*Case 8*

$$x_{11}x_{10}p_6^2 + x_{01}x_{00}p_3^2 + x_{11}x_{01}p_1^{-1}p_6(p_3 - 1) + x_{10}x_{00}p_1p_6(p_3 - 1) + (x_{11}x_{00} + x_{10}x_{01})p_6 \\ + (x_{11}p_6 + x_{10}p_6 + x_{01}p_3 + x_{00}p_3)r_4 + r_4^2 = 0 \qquad (27)$$

also with growth $e_{11}$ in all directions.

*Case 9*

$$(x_{11}x_{00} + x_{10}x_{01}) + p_3(x_{11}x_{01} + x_{10}x_{00}) + p_6x_{11}x_{10} + x_{01}x_{00}p_6^{-1}(p_3 - 1)^2 \\ + r_3(p_6 - p_3 + 1)(x_{01} + x_{00}) + p_6(p_6 + 1)r_3^2 = 0. \qquad (28)$$

This has growth $e_{14}$ in all directions. In the subcase $p_6 = p_3 - 1$ it becomes the integrable case A1 of [8]; on the other hand, if $p_6 \neq p_3 - 1$ it can be translated into the more symmetric form:

$$(x_{11}x_{00} + x_{10}x_{01}) + p_3(x_{11}x_{01} + x_{10}x_{00}) + p_6x_{11}x_{10} + x_{01}x_{00}p_6^{-1}(p_3 - 1)^2 \\ + r_4(x_{11} + x_{10} + x_{01} + x_{00}) + r_4^2 = 0. \qquad (29)$$

*Case 10*

$$(x_{00}x_{10} + x_{01}x_{11}) + (x_{00}x_{11} + x_{01}x_{10})p_3 + (p_3 - 1)(x_{00}x_{01} + x_{10}x_{11}) \\ (x_{00} - x_{01} + x_{10} - x_{11})r_4 + r_4^2 = 0, \qquad (30)$$

also with growth $e_{14}$ in all directions.

*Case 11*

$$(x_{00}x_{11} + x_{01}x_{10} - x_{01}x_{00})p_4 + p_1x_{00}x_{10} + (x_{00}p_1 + x_{10} + x_{11}p_4)r_4 + r_4^2 = 0 \qquad (31)$$

with growths $e_5, e_{16}, e_{16}, e_6$ in the four directions.

*Case 12*

$$x_{00}x_{10}p_1 + x_{01}x_{11}p_3 + x_{00}x_{01}(p_1p_3 - 1) + x_{00}x_{11} + x_{10}x_{01} \\ + (x_{00}p_1 + x_{10} + x_{01}p_3 + x_{11})r_4 + r_4^2 = 0 \qquad (32)$$

with growths $e_5, e_{16}, e_{16}, e_{16}$.

The known models referred to in section 2.3, that have linear factors, all appear in our analysis.

## 5. Some integrable cases

Due to the search condition (one linear factor at (2, 2)), the results are not comprehensive as far as integrable cases are concerned. We can say that they must all be subcases of the cases enumerated above.

Case 2 is integrable as it stands. We can also write it as

$$x_{00}x_{10}c_1 + x_{00}x_{01}c_5 + (x_{00}x_{11} + x_{10}x_{01})c_2 + x_{10}x_{11}c_6 + x_{01}x_{11}c_3 = 0 \qquad (33)$$

and the statement is that this lattice map is integrable for all values of the five parameters $c_i$. If we next consider integrability in the sense of consistency, then the parameters $c_i$ must have specific forms in terms of the spectral parameters associated with the lattice directions. The result is

$$(xx_{[ij]} + x_{[i]}x_{[j]})(\alpha_i - \alpha_j) + xx_{[i]}\big(\beta_j + \alpha_j^2\big)\big/\gamma_j - xx_{[j]}\big(\beta_i + \alpha_i^2\big)\big/\gamma_i$$
$$- x_{[i]}x_{[ij]}\gamma_i + x_{[j]}x_{[ij]}\gamma_j = 0 \qquad (34)$$

where the parameters associated with the third direction are restricted by

$$\begin{vmatrix} 1 & \alpha_1 & \beta_1 \\ 1 & \alpha_2 & \beta_2 \\ 1 & \alpha_3 & \beta_3 \end{vmatrix} = 0. \qquad (35)$$

An immediate way to resolve this constraint is to take

$$\alpha_i = \mu p_i^2 + \nu, \qquad \beta_i = \rho p_i^2 + \sigma.$$

The various homogeneous KdV equations are obtained as subcases (with *one* spectral parameter), the modified KdV of (3) is obtained with $\gamma_i = p_i$, $\rho = -1$, $\mu = \nu = \sigma = 0$ and the Schwarzian KdV of (4) with $\gamma_i = -p_i^2$, $\mu = -1$, $\rho = \nu = \sigma = 0$. Furthermore, the generalized form of [15] is included with $\gamma_i = -(p_i + a)(p_i + b)$, $\mu = -1$, $\nu = 0$, $\rho = -(a^2 + b^2)$, $\sigma = a^2b^2$. By suitable scaling of type $x_{nm} \mapsto A^{n-n_0}B^{m-m_0}x_{nm}$ one can reduce this model to Q3 with $\delta = 0$.

It is perhaps worth mentioning that if some of $c_i$ vanish we may get asymmetric models with different growth in different directions, for example,

$$x_{00}x_{10}c_1 + x_{00}x_{01}c_5 + (x_{00}x_{11} + x_{10}x_{01})c_2 = 0 \qquad (36)$$

has growth patterns $q_2, q_5, q_1, q_5$, and

$$x_{00}x_{10}c_1 + x_{10}x_{11}c_6 + x_{01}x_{11}c_3 = 0 \qquad (37)$$

grows as $q_5, q_4, q_4, q_5$. It is not clear how such special cases carry over to the consistency approach.

As an interesting model worth further study we would like to present (19)

$$(x_{00} - x_{01})(x_{10} - x_{11}) + (x_{00} - x_{11})r_4 + (x_{01} - x_{10})r_3 + u = 0.$$

It is integrable with the growth patterns $q_3$ in all directions. But this model is not symmetric and therefore its consistency formulations are problematic: how should the maps on the other sides of the cube be oriented or do we perhaps need entirely different maps there?

## 6. Conclusions

We have analysed the factorization process for a class of two-dimensional lattice models (quad models). By solving the resulting equations we obtained a list of models. We then conducted an entropy analysis on the results. This provided an ordering among the models, by increasing complexity.

This list of minimally factoring models contains both models with vanishing entropy and models with different nonvanishing entropy. These models with nonvanishing entropy may contain integrable models as subcases, but this has not been comprehensively analysed here.

More can be done along the lines we have followed: one could insist on stronger factorization requirements, or one could start from the most general quartic defining relation, rather than restricting on quadratic ones as we did here. Both of these are beyond the scope of this paper.

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